Web Appendix for Sanford C. Gordon and Dimitri Landa, “Do the Advantages of Incumbency Advantage Incumbents?”

In this appendix, we present equilibrium characterizations for the three models analyzed in the main body of the paper. These characterizations serve, in part, as existence proofs for the nontrivial weak Perfect Bayesian Equilibria we focus on in the paper.

Characterizations of Model Equilibria

Equilibrium in the Campaign Discount Game

We first introduce some useful notation. Let \( t_c(t_i; q, k, d, \hat{w}(\cdot), P_c(\cdot)) \) denote the lowest quality prospective challenger willing to enter the race against an incumbent of type \( t_i \), and \( \bar{t}_c(t_i; q, k, d, \hat{w}(\cdot), P_c(\cdot)) \) the highest quality challenger against whom that incumbent would remain in the race. Likewise, let \( t_i(t_c; q, k, d, \hat{w}(\cdot), P_i(\cdot)) \) denote the lowest quality incumbent willing to stay in the race against a challenger given the challenger’s type \( t_c \), and \( \bar{t}_i(t_c; q, k, d, \hat{w}(\cdot), P_i(\cdot)) \) the highest quality incumbent against whom that challenger would enter the race. (In what follows, we suppress reference in these functions to arguments other than the types of the incumbent and challenger.)

Lemma 1 In equilibrium:

1. the prospective challenger enters if and only if \( t_c > t_i - q + \frac{k}{P_c(\hat{w}(\cdot) - t_i + t_c);} \)
2. if the prospective challenger enters, the incumbent stays in the race if and only if \( t_i > t_c - q + \frac{k-d}{1-P_c(\hat{w}(\cdot) - t_i + t_c);} \)
3. if the race is contested, the voter retains the incumbent if and only if \( w = t_i - t_c + \varepsilon > \hat{w}(\cdot), \) where \( \hat{w}(\cdot) \) is the unique solution to

\[
\int_{-\infty}^{\infty} \int_{t_c(\cdot)}^{\bar{t}_c(\cdot)} t_c p(\tau_i, \tau_c | \hat{w}) d\tau_c d\tau_i = \int_{-\infty}^{\infty} \int_{\bar{t}_i(\cdot)}^{\bar{t}_c(\cdot)} t_i p(\tau_i, \tau_c | \hat{w}) d\tau_i d\tau_c, \tag{W.1}
\]
where $t_i(\cdot), \bar{t}_i(\cdot), t_c(\cdot),$ and $\bar{t}_c(\cdot)$ are defined implicitly by

\[
\begin{align*}
  t_i(\cdot) &= t_c - q + \frac{k - d}{1 - P_\epsilon(\hat{w}(\cdot) - t_i(\cdot) + t_c)}, \\
  \bar{t}_i(\cdot) &= t_c + q - \frac{k}{P_\epsilon(\hat{w}(\cdot) - \bar{t}_i(\cdot) + t_c)}, \\
  t_c(\cdot) &= \bar{t}_i(\cdot) = t_i - q + \frac{k}{P_\epsilon(\hat{w}(\cdot) - t_i + t_c(\cdot))}, \\
  \bar{t}_c(\cdot) &= \bar{t}_i(\cdot) = t_i + q - \frac{k - d}{1 - P_\epsilon(\hat{w}(\cdot) - t_i + \bar{t}_c(\cdot))}, \text{ and} \\
  p(t_i, t_c|w) &= \frac{p_\epsilon(w - t_i + t_c)p_c(t_c)\pi_\epsilon(\bar{t}_i)|\pi_\epsilon|d\bar{t}_i d\bar{t}_c}; \quad \text{(W.2)}
\end{align*}
\]

4. for arbitrary information set $Z$, voter posterior beliefs are given by

\[
\begin{align*}
  p(t_i|Z) &= \frac{\Pr(Z|t_i)p_i(t_i)}{\int_{-\infty}^{\infty} \Pr(Z|\tau)p_i(\tau)d\tau}, \\
  p(t_c|Z) &= \frac{\Pr(Z|t_c)p_c(t_c)}{\int_{-\infty}^{\infty} \Pr(Z|\tau)p_c(\tau)d\tau}; \quad \text{and} \quad \text{(W.3)}
\end{align*}
\]

5. there exist pairs of incumbent and challenger types, \{\(t_i, t_c\)}, that produce contested races if and only if $\hat{w}(\cdot)$ satisfies

\[
q > \frac{k - P_\epsilon(\hat{w}(\cdot) - t_i + t_c)d}{2P_\epsilon(\hat{w}(\cdot) - t_i + t_c)(1 - P_\epsilon(\hat{w}(\cdot) - t_i + t_c))} \quad \text{(W.4)}
\]

and

\[
k < \frac{(2q + d)^2}{8q} \quad \text{(W.5)}
\]

**Proof.** 1. and 2. These are best response correspondences of the challenger and incumbent as described in inequalities (1) and (2) in the text.

3. Equation (W.1) implicitly defines $\hat{w}(\cdot)$, the campaign signal that would induce the voter’s posterior mean on the incumbent to equal that on the challenger. To establish uniqueness, let $w'' > w'$. From (W.2) and the formula for the normal density, $p(t_i, t_c|w'')/p(t_i, t_c|w')$ is strictly increasing in $t_i$ toward $\infty$ and strictly decreasing in $t_c$ toward 0. Therefore, $p(t_i, t_c|w)$ satisfies a strict and unbounded monotone likelihood ratio property (MLRP). MLRP implies that $p(t_i|w'', t_c)$ strictly first order stochastically dominates $p(t_i|w', t_c)$, and $p(t_c|w', t_i)$ strictly first order stochastically
dominates $p(t_c|w'', t_i)$. The MLRP implies that an increase in $\hat{w}$ leads to a strict increase in the left side of equation (W.1) and a strict decrease in the right side. Therefore $\hat{w}(\cdot)$ is unique. Unboundedness of MLRP implies that $\hat{w}(\cdot)$ exists.

4. Equilibrium posterior beliefs are given in (W.3), with $Pr(Z|t_i)$ and $Pr(Z|t_c)$ given by

$$Pr(C = 1, S = 1, w|t_i) = (P_c(\bar{t}_c(\cdot)) - P_c(t_c(\cdot))) \int_{-\infty}^{\infty} p_\varepsilon(w - t_i + \tau_c)p_c(\tau_c)d\tau_c$$

$$Pr(C = 1, S = 1, w|t_c) = (P_i(\bar{t}_i(\cdot)) - P_i(t_i(\cdot))) \int_{-\infty}^{\infty} p_\varepsilon(w - \tau_i + t_c)p_i(\tau_i)d\tau_i$$

$$Pr(C = 1, S = 0|t_i) = 1 - P_c(\bar{t}_c(\cdot))$$

$$Pr(C = 1, S = 0|t_c) = P_i(t_i(\cdot))$$

$$Pr(C = 0|t_i) = P_c(t_c(\cdot))$$

$$Pr(C = 0|t_c) = 1 - P_i(\bar{t}_i(\cdot)).$$

5. For races to be contested, it must be the case that both inequalities (1) and (2) in the text are satisfied. For there to exist pairs $\{t_i, t_c\}$ such that this is the case, the right side of (2) must exceed the right side of (1). Inequality (W.4) expresses this condition solving for $q$ (noting that a contested race implies $\theta^*(t_i, t_c) = 1$). Rearranging (W.4) yields a quadratic inequality in $P_\varepsilon(\hat{w}(\cdot) - t_i + t_c)$. For there to exist values of $\hat{w}(\cdot)$ that satisfy the inequality, the discriminant must be positive; the condition for a positive discriminant is given in Inequality (W.5).

To see that the conjunction of conditions (W.4) and (W.5) is both necessary and sufficient for the possibility of contested races, note that the densities $p_i(t_i)$ and $p_c(t_c)$ are defined over the entire real line. This insures that if these inequalities are satisfied, there will exist pairs of incumbent and challenger types for which an election will be contested.

**Equilibrium in the Endorser Bias Game**

Let $\bar{t}_c(t_i; q, k, d, \hat{w}^i(\cdot), \hat{w}^c(\cdot), P_c(\cdot))$ denote the lowest quality prospective challenger willing to enter the race against an incumbent of type $t_i$, and $\bar{t}_c(t_i; q, k, d, \hat{w}^i(\cdot), \hat{w}^c(\cdot), P_c(\cdot))$ the highest quality challenger against whom that incumbent would remain in the race. Likewise, let $\bar{t}_i(t_c; q, k, d, \hat{w}^i(\cdot), \hat{w}^c(\cdot), P_c(\cdot))$ denote the lowest quality incumbent willing to stay in the race against a challenger given the challenger’s type $t_c$, and $\bar{t}_i(t_c; q, k, d, \hat{w}^i(\cdot), \hat{w}^c(\cdot), P_c(\cdot))$ the highest quality incumbent against whom
Lemma 2 In any equilibrium with an incumbent-biased endorser:

1. endorsement is redundant if and only if \( t_i + b \geq T_c \), where \( T_c \) is defined in part 3 of Lemma 1;

2. if endorsement is redundant, the endorser endorses the incumbent in a contested race, and the equilibrium characterization is otherwise identical to that described in Lemma 1;

3. if endorsement is not redundant,
   
   (a) the prospective challenger enters if and only if \( t_c > t_i - q + \frac{k}{P_c(\hat{w}(\cdot) - t_i + t_c)} \);
   
   (b) if the prospective challenger enters, the incumbent stays in the race if and only if \( t_c \leq t_i + b \);
   
   (c) if the race is contested,
      
      i. the endorser endorses the incumbent if and only if \( t_c \leq t_i + b \) and
      
      ii. the voter elects the challenger with certainty if the endorser endorses the challenger (off the path of play), and otherwise retains the incumbent if and only if \( w = t_i - t_c + \epsilon > \hat{w}(\cdot) \), where \( \hat{w}(\cdot) \) is the unique solution to

      \[
      \int_{-\infty}^{\infty} \int_{t_c(\cdot)}^{t_i+b} \frac{\tau_c p(\tau_i, \tau_c | \hat{w}(\cdot))}{P_c(\tau_c(\cdot)) - P_c(t_c(\cdot))} d\tau_c d\tau_i = \int_{-\infty}^{\infty} \int_{t_i-b}^{\bar{\tau}_i(\cdot)} \frac{\tau_i p(\tau_i, \tau_c | \hat{w}(\cdot))}{P_i(\tau_i(\cdot)) - P_i(t_i(\cdot))} d\tau_i d\tau_c, \tag{W.6}
      \]

      where \( t_c(\cdot), \bar{\tau}_i(\cdot) \), and \( p(t_i, t_c | w) \) are defined as in (W.2);

   (d) voter beliefs on the path of play are derived via Bayes’ Rule as per the system of equations in (W.3), and off the path of play (a contested race in which the incumbent does not receive an endorsement), they may be any posterior beliefs that support \( E[t_i | C = 1, S = 1, B = 0, w] < E[t_c | C = 1, S = 1, B = 0, w] \); and

   (e) there exist pairs of incumbent and challenger types, \( \{t_i, t_c\} \), that produce contested races if and only if \( \hat{w}(\cdot) \) satisfies

   \[
   q > \frac{k}{P_c(\hat{w}(\cdot) - t_i + t_c)} - b. \tag{W.7}
   \]
Proof. We proceed by proving, in order, parts 3(a), 3(c)i, 3(b), 3(c)ii, 3(d), 3(e), 2, and 1. Part 3(a) gives the best response correspondence for the challenger as given in inequality (2) in the text. To see that 3(c)i holds in any equilibrium in which the endorsement is non-redundant, suppose otherwise. Then the endorser sometimes endorses the incumbent if \( t_c > t_i + b \) or the challenger when \( t_c \leq t_i + b \). But, given non-redundancy, this would cause the voter to elect the endorser’s less-preferred candidate with weakly higher probability than if the endorser were playing the strategy in 3(c)i. Thus, the supposition must be be false in any equilibrium in which the endorsement is non-redundant.

We next show that, given 3(c)i, any equilibrium must have the properties described in the remainder of Part 3. Given 3(c)i, an endorsement of the challenger would fully reveal the superiority of the challenger in a contested race, whom the voter would then elect. But then the incumbent would prefer not to stay in the race if challenged. By 3(c)i, the highest quality challenger against whom an incumbent would run has type \( t_i + b \). By monotonicity of the incumbent’s best response correspondence (1), she will also remain in the race against any challenger of lower type; this establishes 3(b). Given 3(a) and 3(b), \( \hat{w}(\cdot) \) is the unique value of \( w \) that would cause the voter to believe \( E[t_c|C = 1, S = 1, B = 1, w] = E[t_i|C = 1, S = 1, B = 1, w] \); equation (W.6) in 3(c)ii expresses this condition in terms of primitives and the strategies of the other players. Uniqueness and existence of \( \hat{w}(\cdot) \) is established using the same logic as in the proof of part 3 of Lemma 1.

3(d). If endorsement is not redundant, equilibrium posterior beliefs on the path of play are evaluated with \( \Pr(Z|t_i) \) and \( \Pr(Z|t_c) \) given by

\[
\begin{align*}
\Pr(C = 1, S = 1, w|t_i) &= (P_c(t_i + b) - P_c(\hat{w}(\cdot))) \int_{-\infty}^{\infty} p_c(w - t_i + \tau_c)p_c(\tau_c)d\tau_c \\
\Pr(C = 1, S = 1, w|t_c) &= (P_i(\hat{r}_i(\cdot)) - P_i(t_i - b)) \int_{-\infty}^{\infty} p_i(w - \tau_i + t_c)p_i(\tau_i)d\tau_i \\
\Pr(C = 1, S = 0|t_i) &= 1 - P_c(t_i + b) \\
\Pr(C = 1, S = 0|t_c) &= P_i(t_i - b) \\
\Pr(C = 0|t_i) &= P_c(\hat{w}(\cdot)) \\
\Pr(C = 0|t_c) &= 1 - P_i(\hat{r}_i(\cdot)).
\end{align*}
\]
If endorsement is redundant, \( \Pr(Z|t_i) \) and \( \Pr(Z|t_c) \) are the expressions given in the proof of Lemma 1 above.

3(e). For contested races to be possible, the challenger’s best response correspondence in part 3(a) and the incumbent’s in part 3(b) must be satisfied. Inequality (W.7) expresses this condition solving for \( q \).

Because the densities \( p_i(t_i) \) and \( p_c(t_c) \) are defined over the entire real line, if this inequality are satisfied, there will exist pairs of incumbent and challenger types for which an election will be contested given a non-redundant endorser.

2. Immediate.

1. To establish that the condition in Part 1 is sufficient, note that for \( t_c(t_i; q, k, d, x_c, x_m, \hat{w}(\cdot), P_\xi(\cdot)) \) as defined in Lemma 1, if \( t_i + b > \overline{t_c}(\cdot) \), the threat of not receiving an endorsement would not deter any incumbents not already deterred given Lemma 1. Because that threat is not binding on any incumbents who would otherwise run, an endorsement for the incumbent is redundant. To see that the condition is necessary, note that if \( t_i + b < \overline{t_c}(\cdot) \), there would exist challengers against whom an incumbent would run in the absence, but not in the presence of endorsements, as established above. But then endorsements would be informative, and thus not redundant.

**Equilibrium in the Partisan Bias Game**

We begin with some useful notation. Let \( t_c(t_i; q, k, d, x_c, x_m, \hat{w}(\cdot), P_\xi(\cdot)) \) denote the lowest quality prospective challenger willing to enter the race against an incumbent of type \( t_i \), and \( \overline{t}_c(t_i; q, k, d, x_i, x_m, \hat{w}(\cdot), P_\xi(\cdot)) \) the highest quality challenger against whom that incumbent would remain in the race. Likewise, let \( \underline{t}_i(t_c; q, k, d, x_i, x_m, \hat{w}(\cdot), P_\xi(\cdot)) \) denote the lowest quality incumbent willing to stay in the race against a challenger given the challenger’s type \( t_c \), and \( \overline{t}_i(t_c; q, k, d, x_c, x_m, \hat{w}(\cdot), P_\xi(\cdot)) \) the highest quality incumbent against whom that challenger would enter the race.

**Lemma 3** In equilibrium:

1. the prospective challenger enters iff \( t_c > t_i - q + (2x_c - 1) + \frac{k}{P_\xi(\hat{w}_m(\cdot)-t_i+t_c)\theta(t_c,t_i)} \);

2. if the prospective challenger enters, the incumbent stays in the race iff \( t_c < t_i + q + (2x_i - 1) - \frac{k-d}{(1-P_\xi(\hat{w}_m(\cdot)-t_i+t_c))} \);
3. if the race is contested, voter \( v \) votes to retain the incumbent if and only if \( w = t_i - t_c + \varepsilon > \hat{w}_v(\cdot) \), where \( \hat{w}_v(\cdot) \) is the solution to

\[
\int_{-\infty}^{\infty} \int_{t_c(\cdot)}^{\overline{t}_c(\cdot)} \frac{\tau_c p(\tau_i, \tau_c | \hat{w}_v)}{P_c(\overline{t}_c(\cdot)) - P_c(t_c(\cdot))} d\tau_c d\tau_i - x_i^2 = \int_{-\infty}^{\infty} \int_{\underline{t}_i(\cdot)}^{\overline{t}_i(\cdot)} \frac{\tau_i p(\tau_i, \tau_c | \hat{w}_v)}{P_i(\overline{t}_i(\cdot)) - P_i(\underline{t}_i(\cdot))} d\tau_i d\tau_c - (1 - x_v)^2, \tag{W.8}
\]

where \( t_c(\cdot), \overline{t}_c(\cdot), \underline{t}_c(\cdot), \) and \( \overline{t}_i(\cdot) \) are defined implicitly by

\[
\begin{align*}
\underline{t}_d(\cdot) &= t_c - q - (2x_i - 1) + \frac{k - d}{1 - P_c(\hat{w}_m(\cdot) - t_i + t_c)}, \\
\overline{t}_d(\cdot) &= t_c + q - (2x_c - 1) - \frac{k}{P_c(\hat{w}_m(\cdot) - t_i + t_c)}, \\
\underline{t}_c(\cdot) &= t_i - q + (2x_c - 1) + \frac{k}{P_c(\hat{w}_m(\cdot) - t_i + t_c)}, \\
\overline{t}_c(\cdot) &= t_i + q + (2x_i - 1) - \frac{k - d}{1 - P_c(\hat{w}_m(\cdot) - t_i + t_c)}, \quad \text{and} \\
p(t_i, t_c | \hat{w}) &= \frac{p_c(\hat{w} - t_i + t_c)p_c(t_c)p_i(t_i)}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_c(\hat{w} - \tau_i + \tau_c)p_c(\tau_c)p_i(\tau_i)d\tau_i d\tau_c}; \tag{W.9}
\end{align*}
\]

4. voter beliefs are derived via Bayes’ Rule as per the system of equations in (W.3); and

5. there exist pairs of incumbent and challenger types, \( \{t_i, t_c\} \), that produce contested races if and only if \( \hat{w}_m(\cdot) \) satisfies

\[
q > (x_c - x_i) + \frac{k}{2(P_c(\hat{w}_m(\cdot) - t_i + t_c))} + \frac{k - d}{2(1 - P_c(\hat{w}_m(\cdot) - t_i + t_c))} \tag{W.10}
\]

and

\[
k < \frac{(2(q + x_i - x_c) + d)^2}{8(q + (x_i - x_c))}. \tag{W.11}
\]

**Proof.** 1. and 2. These are best response correspondences given in inequalities (5) and (6) in the text.

3. Equation (W.8) implicitly defines \( \hat{w}_v(\cdot) \), the campaign signal that would induce voter \( v \) to be indifferent between the incumbent and challenger. Uniqueness of \( \hat{w}_v(\cdot) \) is established using the same logic as in the proof of part 3 of Lemma 1.

4. Equilibrium posterior beliefs are evaluated with \( \Pr(Z|t_i) \) and \( \Pr(Z|t_c) \) identical to those given in Lemma 1, substituting the values of \( t_c(\cdot), \overline{t}_c(\cdot), \underline{t}_c(\cdot), \) and \( \overline{t}_i(\cdot) \) obtained from Part 3 of the
current lemma.

5. For races to be contested, it must be the case that both of the inequalities (5) and (6) are satisfied. For there to exist pairs \( \{t_i, t_c\} \) such that this is the case, the right side of (5) must be greater than the right side of (6) (noting that a contested race implies \( \theta^*(t_i, t_c) = 1 \)). Solving for \( q \), we obtain inequality (W.10). Rearranging (W.10) yields a quadratic inequality in \( P_e(\hat{w}_m(\cdot) - t_i + t_c) \).

For there to exist values of \( \hat{w}_m(\cdot) \) that satisfy the inequality, the discriminant must be positive; the condition for a positive discriminant is given in Inequality (W.11).

To see that the conjunction of conditions (W.10) and (W.11) is both necessary and sufficient for the possibility of contested races, note that the densities \( p_i(t_i) \) and \( p_c(t_c) \) are defined over the entire real line. This insures that if these inequalities are satisfied, there will exist pairs of incumbent and challenger types for which an election will be contested. ■