A New LMM

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Background

- About LMM
  - pros: convenient
    - use of the Black-Scholes model
    - forward measure
  - cons: problems
    - drift freezing
    - forward LIBOR is no longer log-normal
    - no continuous limit (continuous forward rate cannot be log-normal)
Background

- **Forward measure**
  - maturity dependent
  - Jamshidian’s separation theorem (1987)

\[
\mathbb{E}_t \left[ \exp \left( - \int_t^T r(u) du \right) X(T) \right] = \mathbb{E}_t \left[ \exp \left( - \int_t^T r(u) du \right) \tilde{\mathbb{E}}^{(T)}_t [X(T)] \right] = P(t, T) \tilde{\mathbb{E}}^{(T)}_t [X(T)]
\]
A Replacement

• $1+f (=F)$ follows log-normal
  – pros, only pros, no cons
  – consistent with normal short rate models
    • Vasicek and its variations such as
    • Hull-White
  – no need for drift freezing
    • forward measures are deterministically linked
  – same Black-Scholes model for cap/floor
    • call $\rightarrow$ put (Hull)
  – price caps/floors and swaptions simultaneously!
A Replacement

• $1+f (=F)$ follows log-normal
  – closed-form solution to swaptions
    • Jamshidian’s theorem (yet one-factor only)
    • premium quotes anyway (otherwise log-normal volatility would be a problem)
  – approach normal when $f$ is low and maintain shifted log-normal when it is high
  – make perfect economic sense
    • $F$ (forward rate) is reciprocal of $\Psi$ (forward price) which leads to a period-shift in forward measure
The Key is $F$ (forward rate) is reciprocal of $\Psi$ (forward price) which leads to a period-shift in forward measure

- $F(t,T_i,T_j)$ is $T_j$-forward measure
- $\Psi(t,T_i,T_j)$, which is $1/F$ is $T_i$-forward measure

- proof in paper
- it makes perfect sense in that
  - payment on $F$ is paid at $T_j$
  - but this payment can be discounted (since $F$ is known at $T_i$) to $T_i$, which then becomes $\Psi$. 
\( \max \{L_{12} - R^K, 0\} = \max \{F_{112} - R^K, 0\} \)

where \( F_{112} \) is martingale under \( W^2 \)

Note that \( L_{12} \) (and \( F_{112} \)) is known at time \( T_1 \) but used at time \( T_2 \).

\( \frac{1}{K} \max \{K - P_{12}, 0\} \)

\( K = 1 + R^K \)
The Key Equation

\[ d\hat{W}^{(T_j)}(t) = d\hat{W}^{(T_i)}(t) + \xi(t, T_i, T_j)dt \]

where

\[ \xi(t, T_i, T_j) = v(r, t, T_j) - v(r, t, T_i) \]

\[ d\ln P(r, t, T) = r(t) - \frac{v(r, t, T)^2}{2} \ dt + v(r, t, T)d\hat{W}(t) \]

which implies

\[ \xi(t, T_i, T_j) = \xi(t, T_i, T_k) + \xi(t, T_k, T_j) \]

\[ = \xi(t, T_i, T_{i+1}) + \cdots + \xi(t, T_{j-1}, T_j) \]
\[ \frac{dF(t, T_i, T_j)}{F(t, T_i, T_j)} = \xi(t, T_i, T_j) d\tilde{W}^{(T_i)}(t) \]

\[ = \xi(t, T_i, T_j) d\tilde{W}^{(T_i)}(t) + \xi(t, T_i, T_j) dt \]

\[ = \xi(t, T_i, T_j)^2 + \xi(t, T_i, T_j) d\tilde{W}^{(T_i)}(t) \]

\[ = \sum_{k=i}^{j-1} \xi(t, T_k, T_{k+1})^2 + \sum_{k=i}^{j-1} \xi(t, T_i, T_j) d\tilde{W}^{(T_i)}(t) \]
\( \xi \) (The Key Variable)

- instantaneous volatility under forward measure for forward rate
- deterministic
- caplet “put” vol is \( \sqrt{\int_t^{T_j} \xi(u,T_j,T_{j+1})^2 \, du} \) which is just like the BS “price” vol
  - note call on rate = put on price
\( \xi \) (The Key Variable)

- caplet (put) pricing formula

\[
c_{P,j} = \frac{P(t, T_j)}{K} KN \left[ \ln \Psi(t, T_j, T_{j+1}) - \ln K - \frac{v_{P,j}^2}{v_{P,j}} \right] - \Psi(t, T_j, T_{j+1})N \left[ \ln \Psi(t, T_j, T_{j+1}) - \ln K + \frac{v_{P,j}^2}{v_{P,j}} \right]
\]

where (price vol)

\[
v_{P,j}^2 = \tilde{V}[\ln P(r, T_j, T_{j+1})]
\]
\[
= \tilde{V}[\ln \Psi(T_j, T_j, T_{j+1})]
\]
\[
= \int_t^{T_j} \xi(u, T_j, T_{j+1})^2 du
\]
Lognormal vol vs. Normal vol

- volatility of discrete LIBOR rate

\[
\tilde{\V}_t^{(T_{j+1})}[\ell(T_j, T_{j+1})] = \tilde{\V}_t^{(T_{j+1})}[-\ln P(r, T_j, T_{j+1})] \\
= \tilde{\V}_t^{(T_{j+1})}[\ln P(r, T_j, T_{j+1})] \\
= \tilde{\V}_t^{(T_{j})}[\ln P(r, T_j, T_{j+1})] \\
= v_{P,j}^2
\]

\[
\tilde{\V}_t^{(T_{j+1})}[\ell(T_j, T_{j+1})] = e^{2\mu + 2v_{e,j}^2} - e^{2\mu + v_{f,j}^2} \\
= f(t, T_j, T_{j+1})^2 e^{v_{f,j}^2} - 1 \\
\approx f(t, T_j, T_{j+1})^2 (1 + v_{f,j}^2 - 1) \\
= (f(t, T_j, T_{j+1})v_{f,j})^2
\]

\[v_{f,j}^2 = \tilde{\V}_t^{(T_{j+1})}[\ln \ell^*(T_j, T_{j+1})]\]
Lognormal vol vs. Normal vol

Caplet Value and Price Vol
Black Vol = 0.4

alpha 1.2
mu 0.05
sigma 0.1
r0 0.02
Swaption

- **SMM**

  - swap rate \( w \)

\[
w(t, T) = \frac{\sum_{j=1}^{n} P(r, t, T_j) f(t, T_j, T_{j+1})}{\sum_{j=1}^{n} P(r, t, T_j)} = \frac{1 - P(r, t, T_n)}{\sum_{j=1}^{n} P(r, t, T_j)}
\]

- swaption

\[
c_{w,j,n} = \hat{E}_t \left[ \exp \left( - \int_t^s r(u) \sum_{j=1}^{n} P(r, s, T_j \max\{w(s) - w_K, 0\} \right) \right]
\]

\[
= P(t, s) \hat{E}_t^{(s)} \left[ \sum_{j=1}^{n} P(r, s, T_j \max\{w(s) - w_K, 0\} \right]
\]

\[
= P(t, s) \hat{E}_t^{(s)} \left[ \sum_{j=1}^{n} P(r, s, T_j) \hat{E}_t^\Sigma \max\{w(s) - w_K, 0\} \right]
\]

\[
= \sum_{j=1}^{n} P(r, t, T_j) \hat{E}_t^\Sigma \max\{w(s) - w_K, 0\}
\]
Swaption

• SMM
  – SMM and LMM cannot co-exist (Jamshidian)
  – swap rate and LIBOR cannot be simultaneously log-normal
  – proof:
Swaption

- \( E^\Sigma[w(s)] = w(t) \) only if \( n \to \infty \)

\[
\hat{E}_t \left[ \exp \left( -\int_t^s r(u)du \right) \frac{\sum_{j=1}^{n} P(r,s,T_j)}{\sum_{j=1}^{n} P(r,t,T_j)} w(s) \right]
\]

\[
= \hat{E}_t \left[ \exp \left( -\int_t^s r(u)du \right) \frac{\sum_{j=1}^{n} P(r,s,T_j)}{\sum_{j=1}^{n} P(r,t,T_j)} \right] E^\Sigma_t[w(s)]
\]

\[
= \hat{E}_t^\Sigma[w(s)]
\]

\[
\hat{E}_t \left[ \exp \left( -\int_t^s r(u)du \right) \frac{\sum_{j=1}^{n} P(r,s,T_j)}{\sum_{j=1}^{n} P(r,t,T_j)} w(s) \right]
\]

\[
= w(t) - \frac{1 - P(r,t,s)}{\sum_{j=1}^{n} P(r,t,T_j)}
\]

\[
\approx w(t)
\]
Swaption

- Pricing formula
  - swap P&L
    \[ V(u) = (w(u) - w(t)) \sum_{j=1}^{n} P(r, u, T_j) \]
    \[ = \text{floating rate bond} - \text{fixed rate bond} \]
    \[ = 1 - B(u, T; w_t) \]
  - swaption payoff
    \[ \max\{w(u) - w_K, 0\} \sum_{j=1}^{n} P(r, u, T_j) = \max\{1 - B(u, T; w_K), 0\} \]
  - Jamshidian Theorem (breaking up K)
    \[ c_{B,j,n} = \max \ w_K \sum_{j=1}^{n} K_j + K_n - w_K \sum_{j=1}^{n} P(r, u, T_j) + P(r, u, T_n), 0 \]
    \[ = \sum_{j=1}^{n} w_K \max K_j - P(r, u, T_j), 0 + \max\{K_n - P(r, u, T_n), 0\} \]
Swaption

- Pricing formula

\[ c_{B,j,n} = \sum_{j=1}^{n} w_{K} x_{j} + x_{n} \]

\[ x_{j} = P(r, t, T_{j}) K_{j} N \left( \frac{\ln \Psi(t, T_{j}, T_{j+1}) - \ln K - \frac{\sigma_{P,j}^{2}}{v_{P,j}}}{v_{P,j}} \right) - \Psi(t, T_{j}, T_{j+1}) N \left( \frac{\ln \Psi(t, T_{j}, T_{j+1}) - \ln K + \frac{\sigma_{P,j}^{2}}{v_{P,j}}}{v_{P,j}} \right) \]

- swaption is price quote, not vol quote
- so no reason to do SMM
- however, not suitable for multiple factors
  - use it anyway as an approximation (Chen-Scott)
The Perfect Calibration

• caplet is a special swaption: a into b
  – $1 \times 1$ swaption (=caplet)
    \[
    \int_t^{T_1} \xi(u, T_1, T_2)^2 \, du = \xi_{012}^2 \Delta_1 = x_{11}^2
    \]
  – $1 \times 2$ swaption
    \[
    \int_t^{T_1} \xi(u, T_1, T_2)^2 \, du = \xi_{012}^2 \Delta_1 = x_{11}^2
    \]
    \[
    \int_t^{T_1} \xi(u, T_1, T_3)^2 \, du = \int_t^{T_1} \{\xi(u, T_1, T_2) + \xi(u, T_2, T_3)\}^2 \, du
    \]
    \[
    = \{\xi_{012} + \xi_{023}\}^2 \Delta_1
    \]
  – $2 \times 1$ swaption
    \[
    \int_t^{T_2} \xi(u, T_2, T_3)^2 \, du = \int_t^{T_1} \xi(u, T_2, T_3)^2 \, du + \int_{T_1}^{T_2} \xi(u, T_2, T_3)^2 \, du
    \]
    \[
    = x_{12}^2 + x_{22}^2 = \xi_{023}^2 \Delta_1 + \xi_{123}^2 \Delta_2
    \]
The Perfect Calibration

- caplet is a special swaption: a into b
Simulation

- different from LMM
- no approximation
- simulate all forward measures at all times
  - okok... this is a con. but this is error-free (as opposed to LMM where errors accumulate over time) and as a result, our model can price VERY LONG swaptions and exotics
  - vendors (e.g. Dxxx and Nxxx) cannot price swaptions accurately over 10 years.
Simulation

- Although measures are linked, but realistically for each discounting, only the corresponding (forward) measure can be used.
- Hence, cannot discount along path
  - Cash flows, yes.
  - Discounting can only be “outside” of MC.
Extensions

• Thanks to (log)normality, really straightforward
  – to multiple factors
    • e.g. CMS
  – to multiple curves
    • e.g. FX
  – no need to change any structure of single factor
  – a newly added layer (ideal for software engineering)
Extensions

• Need to think about more flexibility between normal and log-normal
  – currently no flexibility
  – could lose model-free result
Stochastic $\xi$

- **Example**
  - **Vasicek** (everything works out)

\[
v^2_{P,j} = \left( \frac{1 - e^{-\alpha(T_{j+1}-T_j)}}{\alpha} \right)^2 \sigma^2 \frac{1 - e^{-2\alpha(T_j-t)}}{2\alpha}
\]

\[
= \int_{t}^{T_j} \xi(u,T_j,T_{j+1})^2 du
\]

- for caplets only, piece-wise flat $\alpha$
  - **Cox-Ingersoll-Ross** (nothing works out)

- volatility is not linear anymore
- forward measures still valid yet not helpful
- simulations still okay but no calibration
Conclusion

• Solve LMM problem by letting $1+L$ be lognormal
• Exact drift adjustment
• Perfect calibration of caps and swaptions