Optimal Investment and Pricing in the Presence of Defaults

Scott Robertson
Questrom School of Business, Boston University

Joint Work with Tetsuya Ishikawa
Morgan Stanley

ECMF
November 5, 2017
Research Goals

Solve the optimal investment problem when the underlying traded asset may default.

- Price defaultable bonds.
- Price dynamic default insurance.

Obtain explicit answers.

- Provide a PDE counterpart to the BSDE pricing literature.
Motivation

Say our goal is to price a claim whose payoff is contingent upon survival of a reference entity.

- Payoff: $\phi_T 1_{T>\delta}$
- $\delta$: default time of a firm $S$.

In practice, pricing is done under a risk neutral measure.

Two problems:

- What risk neutral measure?
- What is the underlying traded asset? What if the underlying is the reference entity?
Motivation

Say our goal is to insure ourselves against losses from the default of a stock in which we own a position.

We could enter into a CDS

- What if investment horizon does not match CDS maturity?
- What if we want dynamic protection?

Is there a fair price for dynamic protection taking into account market incompleteness, and our preferences?
Optimal investment and indifference pricing with defaults have been extensively studied.

- Primarily from the "BSDE" perspective, especially with respect to pricing.

- We fill in a gap by considering Markovian factor models, using PDE techniques, and focusing on indifference pricing.

  - Amenable to computation and analysis.

The computation of dynamic default insurance has been much less well-studied.
Contribution to the Literature

(selected) "PDE" articles

- [Lin06]: Merton model with default intensity $\gamma_t = \gamma(S_t)$ under a fixed risk neutral measure. Analytical formulas for European option prices.

- [SZ07]: single stock factor model similar to ours. However, investor does not lose money in stock upon default.

- [BBC16]: risk-sensitive control problem in factor model with multiple securities, default state dependent intensities. Investor does not lose money in stock upon default.

- [BC16]: optimal investment/consumption problem for power utility in a factor model with multiple securities, default state dependent intensities. Investor loses money upon default.
Contribution to the Literature

(selected) "BSDE" articles

- [Mor09, LQ11]: single stock and non-traded claim. Brownian setting prior to default.

- [JP11, JKP13]: single/multiple stocks along with claim. Multiple credit events which cause a jump in stock prices with trading possible after jump. Brownian setting

- [MS17]: stock modeled as a pure-jump Levy process.

- [LQ15]: extension of [LQ11] to partial information models.

- [GN15, CGN15]: mean-variance hedging under default risk.
Model

Reduced form, "hybrid" intensity model: [SZ07].

$X$: underlying factor process

- $dX_t = b(X_t)dt + a(X_t)dW_t$.
- Solution to Martingale problem for $L$ on $E \subseteq \mathbb{R}^d$ where
  - $L = \frac{1}{2} \text{Tr}(AD^2) + b'\nabla$
  - $E = \bigcup_n E_n$ with $E_n$ bounded, $E_n \uparrow$, $\partial E_n$ smooth.

One risky asset $S$ (riskless asset set to 1)

- $S$ defaults at the random time $\delta$. Prior to $\delta$, $S$ has instantaneous returns, variances, correlations driven by $X$. 
Model

Start at $t \geq 0$. $X^{t,x}_t = x \in E$. Write $X = X^{t,x}$.

$$\frac{dS_s}{S_s} = 1_{s \leq \delta} \left( (\mu - \gamma)(X_s) ds + (\sigma \rho)(X_s)' dW_s + \left( \sigma \sqrt{1 - \rho' \rho} \right)(X_s) dW_s^0 \right)$$

$$- dM_s; \quad s \geq t.$$

- $W^0$: one-dim B.M. $\perp \perp$ of $W$.
- $\delta := \inf \{ s > t : \int_t^s \gamma(X_u) du = -\log(U) \}$, $U \perp \perp W, W^0$.
- $H_s := 1_{s \geq \delta}$; $M_s := H_s - \int_t^{s \wedge \delta} \gamma(X_u) du$.
- $G := \mathcal{F}^{W, W^0} \vee \mathcal{F}^{H}$. $W, W^0, M$ are $G$ local martingales.
- $\mu, \sigma, \gamma, \rho$ smooth functions on $E$, $\gamma, \sigma > 0$, $\rho' \rho \leq 1$. 
Optimal Investment Problem

Investment horizon: \([t, T]\) for \(T > t\).

\(\mathcal{M}\): equivalent local martingale measures on \(\mathcal{G}_T\). \(\widetilde{\mathcal{M}}\) subset with finite relative entropy w.r.t. \(\mathbb{P}\).

\(\mathcal{A}\): acceptable (dollar) trading strategies \(\pi\).

- Wealth process \(\mathcal{W}^{\pi, w} = w + \int_t^T \pi_u dS_u / S_u \).
- Dollar position \(\pi_\delta\) lost at \(\delta\).

\(\pi \in \mathcal{A}\) if \(\mathcal{W}^{\pi, w}\) is a \(\mathbb{Q}\) local martingale for all \(\mathbb{Q} \in \widetilde{\mathcal{M}}\).
Optimal Investment Problem

Exponential investor: \( U(w) := -e^{-\alpha w}, w \in \mathbb{R} \).

Investor

- Trades in \( S \) according to \( \pi \in A \).
- Owns a non-traded claim with time \( T \) payoff \( \phi(X_T)1_{T<\delta} \).
  - \( \phi \) smooth, bounded. Primarily care about \( \phi \equiv 1, \phi \equiv 0 \).

For 0 initial wealth write \( \mathcal{W}^\pi = \mathcal{W}^{\pi,0} \) and define

\[
 u(t, x; \phi) := \sup_{\pi \in A} E \left[ -e^{-\alpha (\mathcal{W}^\pi_T + \phi(X_T)1_{\delta>T})} \right]; \quad (X_t = x)
\]

\[
 G(t, x; \phi) := -\frac{1}{\alpha} \log \left( -u(t, x; \phi) \right).
\]
\[ G(t, x; \phi) = -\frac{1}{\alpha} \log (-u(t, x; \phi)) \]: Certainty Equivalent

Heuristics using DPP suggest \( G \) should solve

\[
0 = G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right);
\]

\[ \phi = G(T, \cdot) \]

\[ \theta(y) \text{: inverse of } ye^y \text{ and } \theta_G := \theta \left( \frac{\gamma}{\sigma^2} e^{\frac{\mu}{\sigma^2}} + \alpha G - \frac{\alpha}{\sigma} \nabla G' a \rho \right). \]

If \( G \) is a classical solution, DPP suggests optimal strategy is

\[ \hat{\pi}_s = \hat{\pi}(s, X_s^{t,x}) \text{ for } \hat{\pi} = \frac{1}{\alpha} \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho - \theta_G \right). \]
Certainty Equivalent PDE

\[ 0 = G_t + LG - \frac{\alpha}{2} \nabla G'A \nabla G + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G'a \rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right); \]
\[ \phi = G(T, \cdot) \]

- This is a semi-linear degenerate parabolic PDE.
  - Non-linearities arise due to market incompleteness.
- Luckily: \( \theta(y) \approx \log(y) - \log(\log(y)), y \gg 0. \)
  - PDE is quadratically growing in \( G, \nabla G. \)
- Regarding solutions/verification:
  - For general regions \( E, \) local ellipticity, verification is hard:
    lack gradient estimates near \( \partial E. \)
  - We must enforce some additional (global) condition.
The Main Assumption

Set $\ell := (\mu - \gamma)/\sigma$ (market price of risk).

Today: assume "strictly incomplete" market absent default.

- The paper treats the "complete" case as well.

Main assumptions:

- $\sup_{x \in \mathbb{E}} \rho'(x) < 1$.
- For some $\varepsilon > 0$ we have for each $n$

$$
\sup_{x \in \overline{E}_n} E^x \left[ e^{\varepsilon \int_0^T \ell(X_u)^2 du} \right] = C(\varepsilon, n) < \infty.
$$

This assumption is MILD. Holds in virtually all models.

- E.g. $X \sim OU, CIR, \mu, \sigma^2, \gamma$ affine.
The Main Result

\[ 0 = G_t + LG - \frac{\alpha}{2} \nabla G'A\nabla G + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G'a\rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta^2_G - 2\theta_G \right); \]

\[ \phi = G(T, \cdot) \]

**Theorem:** assume \( \sup_{x \in E} \rho'(x) < 1 \) and for some \( \varepsilon > 0 \):

\[ \sup_{x \in E} E_x^x \left[ e^{\varepsilon \int_0^T \ell(X_u)^2 du} \right] = C(n) < \infty, \forall n. \]

Then

- The certainty equivalent \( G \) is a classical \((C^1, 2)\) solution.
- The optimal trading strategy is
  \[ \hat{\pi}_s = \hat{\pi}(s, X_{s}^t, x) \text{ for } \hat{\pi} = \frac{1}{\alpha} \left( \frac{\mu}{\sigma} - \frac{\alpha}{\sigma} \nabla G'a\rho - \theta_G \right). \]
- The optimal martingale measure \( \hat{Q} \) has density
  \[ \hat{Z}_s = e^{-\alpha \left( W_s^\hat{\pi} - G(t, x; \phi) + 1_{\delta > s} G(x, X_{s}, \phi) \right)}. \]
Application: Pricing for Defaultable Bonds

Investor owns $q$ units notional: claim payoff $q1_{\delta > T}$.

(Per-unit, buyer’s) indifference price: $p(t, x; q)$ solving

- $u(t, x; 0, 0) = u(t, x; q, -qp(t, x; q)) = e^{\alpha qp(t, x; q)}u(t, x; q, 0)$.

- $u(t, x; \phi, w)$: utility for initial wealth $w$.

- Well known $p$ does not depend on $w$.

Immediate result as $G(t, x; q) = -(1/\alpha)\log(-u(t, x; q))$:

- $p(t, x; q) = \frac{1}{q} (G(t, x; q) - G(t, x; 0))$. 
Application: Dynamic Default Insurance

Goal: find a fair price for dynamic protection against default.

- Approximation to CDS pricing valid for frequent contract adjustments.

Motivation from [SZ07]: optimal investment/pricing but with no loss at default.

- $\pi_\delta$ not lost at default time $\delta$.

How is this possible? What contact has been entered into which enables this?
Dynamic Default Insurance

Perspective: investor has two alternatives:

- A) Do not purchase protection. Lose $\pi_\delta$ at $\delta$. Indirect utility of $u(t, x)$.
- B) Purchase protection. Pay a (per-unit) cash flow rate of $f$, where $f$ is to-be-determined.

- Wealth dynamics:

\[
d\mathcal{W}_{\pi,d}^s = \pi_s 1_{s \leq \delta} \left( (\mu - \gamma)(X_s) - f_s \right) ds \\
+ \pi_s 1_{s \leq \delta} \left( (\sigma \rho)(X_s)'dW_s + (\sigma \sqrt{1 - \rho' \rho})(X_s)dW^0_s \right).
\]

- Indirect utility

\[
u^d(t, x) := \sup_{\pi \in \mathcal{A}_d} E \left[ -e^{-\alpha \mathcal{W}_T^{\pi,d}} \right].
\]
Dynamic Default Insurance

\[ G(t, x) = -\frac{1}{\alpha} \log(-u(t, x)) ; \quad G^d(t, x) = -\frac{1}{\alpha} \log(-u^d(t, x)) . \]

Guess \( f_t = f(t, X_t) \). Find \( f \) so that PDEs for \( G, G^d \) are the same (both have terminal condition \( \phi \)).

\[ 0 = G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G \]
\[ + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right) ; \]
\[ 0 = G^d_t + LG^d - \frac{\alpha}{2} \nabla (G^d)' A \nabla G^d \]
\[ + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu - f}{\sigma^2} - \frac{\alpha}{\sigma} \nabla (G^d)' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} \left( 1 - e^{\alpha G^d} \right) \right) . \]
Dynamic Default Insurance

Upon inspection, given a solution $G$ to the first PDE, $G$ will solve the second PDE if $f$ satisfies

\[
\frac{f_\pm}{\sigma^2} = \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \pm \sqrt{\left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 - \left( \theta_G^2 + 2\theta_G - \frac{2\gamma}{\sigma^2} e^{\alpha G} \right)}.
\]

- Term inside square root is non-negative: real solutions.
- We choose the "-" solution.
  - Lowest possible $f$ since this is what the investor pays.
  - Can also justify $f_-$ by inspecting optimal strategies $\pi^d_\pm$: $f_+ > 0$ and $\pi^d_+ < 0$ - not feasible.
Dynamic Default Insurance

We define the dynamic default insurance protection price

\[ f := \sigma^2 \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho - \sqrt{\left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 - \left( \theta_G^2 + 2\theta_G - \frac{2\gamma}{\sigma^2} e^{\alpha G} \right)} \right). \]

Facts

- \( f \leq \gamma e^{\alpha (G + \hat{\pi})} = \gamma \hat{Q} \): the default intensity under the dual optimal measure \( \hat{Q} \).
  - Equality only when \( \hat{\pi} = 0 \).

- \( f > 0 \) when \( \hat{\pi} > 0 \): intuitive. Pay for protection when long.

- \( f > 0 \) possible even when \( \hat{\pi} < 0 \), but \( f < 0 \) for \( \hat{\pi} \ll 0 \).
Application: $X \sim CIR$, affine market price of risk.

- $dX_t = \kappa (\theta - X_t) dt + \xi \sqrt{X_t} dW_t$.

- Prior to default

  - $dS_t/S_t = \mu X_t dt + \sigma \sqrt{X_t} \left( \rho dW_t + \sqrt{1 - \rho^2} dW^0_t \right)$.

- Default intensity: $\gamma_t = \gamma X_t$.

Assume $\mu \in \mathbb{R}, \sigma, \gamma > 0$ and $|\rho| < 1$.

- Main assumption holds provided $\kappa \theta > \xi^2/2$. 
Investor owns $q$ units of a defaultable bond.

$p(0, x; q)$ as a function of $q, x$ for $T = 1$.

- Physical default prob of 3% at $x = 6\%$ (long run mean).
- $q = 1$ (dash), $q = 3$ (dot-dash), $q = 5$ (dot), $q = 10$ (solid).
Application: Dynamic Default Insurance

\( f(0, x) \) as a function of \( x \) for \( T = 1 \).

- \( \gamma^\hat{Q} \) (dash), \( f \) (solid), \( \gamma \) (dash).
THANK YOU!
John R Birge, Lijun Bo, and Agostino Capponi, *Risk sensitive asset management and cascading defaults.*

Lijun Bo and Agostino Capponi, *Portfolio choice with market-credit risk dependencies.*


Marie-Amelie Morlais, Utility maximization in a jump market model, Stochastics 81 (2009), no. 1, 1–27. MR 2489997
