A Result in Complex Series Expansion

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Abstract

In this presentation, we aim to give a theoretically interesting result that could have many applications. To motivate the discussion, observe the following identity

\[ \sum_{n=0}^{\infty} p(n)z^n = q(z)(1 - z)^{-(k+1)} \]  

(†)

Where \( p(n) \) and \( q(z) \) are both polynomials of degree \( k \). At first sight, one might be misled about the difficulty of this problem since we have polynomials of the same order on both sides. However, a more careful inspection reveals that on the one hand \( p \) varies with the index \( n \), and on the other hand \( q \) varies with a complex variable \( z \). Also the LHS has \( z \) to the power \( n \) whereas the RHS has \( (1 - z) \) to the power \( -(k+1) \).

Significance of Result

If proven true, (†) says that given an arbitrary complex power series with coefficients \( c_n = p(n) \) we can always find a \( q(z) \) such that the identity holds. In fact our proof shows that this implication goes both way. We posit that any choice of \( p(n) \) uniquely determines a \( q(z) \) such that (†) holds, and vice versa.

Statement of the Problem

Let \( p \) be a polynomial of degree \( k > 0 \). Prove that \( \sum p(n)z^n \) has radius of convergence 1 and that there exists a polynomial \( q(z) \) of degree \( k \) such that

\[ \sum_{n=0}^{\infty} p(n)z^n = q(z)(1 - z)^{-(k+1)} \quad (|z| < 1). \]

Details of the Proof

Proof. We first note that any polynomial \( p \) of degree \( k \) can be written as

\[ p = a_0 + a_1n + \cdots + a_kn^k \]

Then we can apply the ratio test to the series \( \sum p(n)z^n \) since

\[
\frac{p(n+1)z^{n+1}}{p(n)z^n} = \frac{a_0 + a_1(n+1) + \cdots + a_k(n+1)^k}{a_0 + a_1n + \cdots + a_kn^k} \cdot \frac{z}{z} \\
= \frac{a_0(n+1)^k + a_1(n+1)^{k+1} + \cdots + a_k}{a_0(n+1)^k + a_1(n+1)^{k+1} + \cdots + a_k} \cdot \frac{z}{z}
\]

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Now as $n \to \infty$, every term in the fraction vanishes except for $a_k$ in the numerator and $a_k n^k (n+1)^{-k}$, which converges to $a_k$. This is evident from elementary limit rules since the vanishing terms all have the form of a rational function whose denominator is of higher order in $n$ than the numerator.

$$\lim_{n \to \infty} \left| \frac{p(n+1)z^{n+1}}{p(n)z^n} \right| = |z|$$

Therefore we conclude $\sum p(n)z^n$ has radius of convergence 1.

Existence of the Polynomial $q(z)$

We first find a series expansion for $(1 - z)^{-(k+1)}$

We successively differentiate $f = (1 - z)^{-1}$

$$f' = (1 - z)^{-2}$$
$$f'' = 2(1 - z)^{-3}$$
$$f^{(3)} = 3 \cdot 2(1 - z)^{-4}$$
$$\vdots$$
$$f^k = k!(1 - z)^{-(k+1)}$$

Then since we can expand $f$ in its series form and obtain the series expression for $f^k$, we have

$$k!(1 - z)^{-(k+1)} = f^k = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)z^{n-k}$$

$$\implies (1 - z)^{-(k+1)} = \frac{1}{k!} \sum_{n=k}^{\infty} \underbrace{n(n-1) \cdots (n-k+1)}_{\text{k terms}} z^{n-k}$$

$$= \frac{1}{k!} \sum_{n=0}^{\infty} (n+k)(n+k-1) \cdots (n+1)z^n$$

$$= \frac{1}{k!} \sum_{n=0}^{\infty} (n^k + c_k n^{k-1} + \cdots + c_1 n + c_0) z^n$$

And we can express this as a sum of convergent series because for $|z| < 1$ the above series is convergent

$$(1 - z)^{-(k+1)} = \frac{c_0}{k!} \sum_{n=0}^{\infty} z^n + \cdots + \frac{c_{k-j}}{k!} \sum_{n=0}^{\infty} n^{k-j} z^n + \cdots + \frac{1}{k!} \sum_{n=0}^{\infty} n^k z^n \quad (*)$$

Next we express the left hand sum as

$$\sum_{n=0}^{\infty} p(n)z^n = \sum_{n=0}^{\infty} (a_0 + \cdots + a_k n^k) z^n$$
Since for $|z| < 1$ we have proven that this series converges, we can write it as a sum of convergent series with coefficients $a_i n^i$ for $i \in (0, \ldots, k)$

$$
\sum_{n=0}^{\infty} p(n)z^n = a_0 \sum_{n=0}^{\infty} z^n + \cdots + a_k \sum_{n=0}^{\infty} n^k z^n
$$

Then define a polynomial $q(z)$ of order $k$ in $z$ by the following expression

$$
\frac{q(z)}{(1 - z)^{-(k+1)}} = \frac{b_0}{1 - z} + \frac{b_1}{(1 - z)^2} + \cdots + \frac{b_k}{(1 - z)^{-(k+1)}} \tag{**}
$$

Evidently $q(z)$ is of order $k$ because by multiplying through $(1 - z)^{-(k+1)}$ we find the RHS’s highest term is $b_0(1 - z)^k$. Therefore we fix the polynomial $q(z)$ by specifying the parameters $b_i$ for $i \in (0, \ldots, k)$.

**Fitting the Parameters** $b_i$ Next we use (*) to obtain a series expression for (**)

$$(**)=b_0 \sum_{n=0}^{\infty} z^n + \cdots + b_i \left( \frac{c_0}{i!} \sum_{n=0}^{\infty} z^n + \cdots + \frac{c_i-j}{i!} \sum_{n=0}^{\infty} n^{i-j} z^n + \cdots + \frac{1}{i!} \sum_{n=0}^{\infty} n^i z^n \right)

+ \cdots + b_k \left( \frac{c_0}{k!} \sum_{n=0}^{\infty} z^n + \cdots + \frac{c_k-j}{k!} \sum_{n=0}^{\infty} n^{k-j} z^n + \cdots + \frac{1}{k!} \sum_{n=0}^{\infty} n^k z^n \right)

$$

Then we have acquired a system of $k + 1$ linear equations in $k + 1$ unknowns $b_m$ for $m \in (0, \ldots, k)$ since we can group (**)

by

$$
a_0 \sum_{n=0}^{\infty} z^n = b_0 \sum_{n=0}^{\infty} z^n + \cdots + b_i \frac{c_0}{i!} \sum_{n=0}^{\infty} z^n + \cdots + b_k \frac{c_0}{k!} \sum_{n=0}^{\infty} z^n

\vdots

a_k \sum_{n=0}^{\infty} n^k z^n = \frac{b_k}{k!} \sum_{n=0}^{\infty} n^k z^n

\Rightarrow a_0 = b_0 + \cdots + b_i \frac{c_0}{i!} + \cdots + b_k \frac{c_0}{k!}

\vdots

a_k = \frac{b_k}{k!}

$$

Which has a solution since we can back substitute through $b_k = k! a_k$. Whence we conclude that there exists a $k$-th degree polynomial $q(z)$ satisfying $q(z)(1 - z)^{-(k+1)} = \sum_{n=0}^{\infty} p(n)z^n$ for any $p(n)$ of order $k!$